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Fin Methods of Higher Order

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The classical method which describes heat conduction in elongated bodies, e g, fins, is extended to methods of higher order Numerical examples are put forward to show that these higher order methods are capable of yielding accurate results with relatively little effort A comparison with a classical method of the finite-difference variety shows that these methods may sometimes be superior It is expected that they may be applied with the same accuracy to fields other than heat transfer Fluid mechanics and elasticity are likely candidates. It is shown that higher order fin methods can easily be applied to nonlinear problems On the other hand, a successful application of the methods requires smooth boundary conditions Any feature in the problem definition that would lead to local singularities, such as sharp corners or abrupt changes in the boundary conditions, renders these methods less effective Therefore, paradoxically, these higher order "fin methods" do not lend themselves very well for the derivation of more accurate solutions in the case of actual cooling fins Nevertheless, since the methods are based on the original treatment of heat transfer in fins, and for want of a better terminology, it would seem appropriate to call then fin methods

1. INTRODUCTION

Cooling fins are thin good heat-conducting plates or spines projecting from a body which produces large amounts of heat. These fins are used to considerably increase the surface of such bodies, without adding greatly to their bulk. In the mathematical treatment of the conduction of heat through these fins [1–3] a method is used which assumes the temperature to be constant within every cross section. This "fin method" involves basically an integration of the heat conduction equation across the fin, upon which the heat-transfer function which is valid at the outside of the fin can be entered quite naturally. As such, the fin method expresses an exact balance between longitudinal conduction and heat transfer through the side walls of the fin. Put in mathematical terms this means a reduction from a two-dimensional model to a one-dimensional one, which is a considerable simplification.

The fin method works quite well if the variation of the temperature in the

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lengthwise direction occurs over distances that are relatively long in comparison with the width of the fin. In that case the two-dimensional effects that occur at the two ends of the fin are negligible, except, of course, in those cases where an accurate description is needed precisely at (one of) these ends [4, 5]. Should the longitudinal temperature variation become more pronounced, e.g., owing to a more rapid variation in heat transfer from the side walls, the original fin method may become too inaccurate. In order to preserve the ease of calculation which makes the fin method so attractive, it would seem to be worthwhile to see whether it is possible to develop higher order fin methods with which we shall be able to handle these more extreme cases with some degree of accuracy.

Another reason why such higher order methods would be a welcome extension of the original fin concept is that it is sometimes necessary to know the variation of the temperature across the fin. This is not true so much for actual cooling fins, as an appreciable variation of the temperature across then would mean a less effective performance in relation to their bulk. However, important examples can be found in crystal growth. In the so-called Bridgman–Stockbarger technique an elongated, often closed, crucible is used to melt and recrystallize a crystal inside. By some means heat is injected into the crucible through its outer wall, quite often by a heat source of restricted size which surrounds part of the crucible. The heat is conducted into the crucible-crystal system and then away from the heat source in the two axial directions, before it is transmitted to the surroundings through the crucible walls at either side of the heat source. When the crystal is melted, it is often necessary to remain within a few degrees of the melting point. However, quite frequently the temperature variation from the outer wall to the crystal core is much larger than this limited temperature range. Therefore, although they have been proposed here too [6], the original fin method would seem to be insufficient for these not-so-slender heat-transfer systems.

Of course, finite-element or finite-difference methods can be applied successfully in these more complicated cases. Examples can be found in the crystal-growth literature [7]. However, the application of these methods leads to lengthy and time-consuming computer codes which are suitable for preliminary in-depth investigations into the field. They appear to be less suitable for real-time control of an actual crystal-growth system. It is for this reason that we hope to be able to develop faster methods based on the fin concept.

It is the purpose of this paper to investigate the performance of higher order fin methods for some simple heat-transfer examples that can also be solved exactly. This will enable us to assess their accuracy. However, the applicability seems by no means restricted to problems in heat transfer. It seems likely that higher order fin methods can be used with success in such diverse fields as fluid mechanics or elasticity. On the other hand, it will be clear that the class of methods we present here is suitable only for problems which involve smooth geometries and smooth boundary conditions. Any irregularities in the field produce local two-dimensional effects, very often singularities, which cannot be modeled effectively with the simple concepts of the fin methods. What one could do is to consider these irregular behaviors within a framework as put forward in [4]. Outside a limited region around an irregular point the field is smooth again, and fin methods will be applicable. Within the restricted region a full two-dimensional approach is necessary. A more comprehensive discussion of this matter is outside the scope of this paper.

The classical fin is irregular in the above sense, since it features sharp corners where it projects from the main body. Therefore, without the special precautions mentioned above, the higher order methods proposed here do not seem to work very well for actual fins. This circumstance would seem to turn the concept of a higher order fin method into something of a misnomer. However, since these methods are direct extensions of the original fin method, and for want of a better terminology, we nevertheless propose to designate them as such, although their application will be found mainly in other fields.

2. A Hierarchy of Fin Methods

An easy way to introduce these methods is to show how they work in the case of a simple model example. Let us consider Laplace's equation

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0 \tag{1}$$

in the domain $0 \leqslant x < \infty$ and $0 \leqslant y \leqslant 1$. Boundary conditions are chosen so that we may readily obtain an analytical solution with which to compare our approximate results. We have

$$\frac{\partial T}{\partial y}=0 \qquad \text{at} \quad y=0 \tag{2}$$

$$\frac{\partial T}{\partial y}=-\varepsilon T+q(x) \qquad \text{at} \quad y=1 \tag{3}$$

$$\frac{\partial T}{\partial x}=0 \qquad \text{at} \quad x=0 \tag{4}$$

$$T \to 0 \qquad \text{as} \quad x \to \infty. \tag{5}$$

This system represents heat conduction in a slab with a heat-input function $q(x)$ at the boundary $y=0$ and linearized heat radiation to the surroundings. For our explicit numerical example we shall use a Gaussian distribution

$$q(x)=\alpha^{1/2}e^{-\alpha x^2} \tag{6}$$

The analytical solution can then easily be obtained by means of the Fourier transform

$$T=\frac{1}{\pi^{1/2}}\int_0^\infty \frac{\cosh(\omega y)\cos(\omega x)}{\omega \sinh \omega + \varepsilon \cosh \omega}\, e^{-\omega^2/4\alpha}\, d\omega. \tag{7}$$

Zero-Order Fin Method

In the classical fin-model approach the temperature T is assumed to be constant across the slab. Equation (1) is then integrated across the slab from $y=0$ to $y=1$. This yields

$$\frac{d^2T}{dx^2}+\left[\frac{\partial T}{\partial y}\right]_{y=0}^{y=1}=0. \tag{8}$$

Together with Eqs. (2) and (3) this yields an equation for a one-dimensional temperature field:

$$\frac{d^2T}{dx^2}-\varepsilon T+\alpha^{1/2}e^{-\alpha x^2}=0. \tag{9}$$

The solution to this equation which satisfies (4) and (5) is

$$T=\frac{1}{4}\left(\frac{\pi}{\varepsilon}\right)^{1/2} e^{\varepsilon/4\alpha}\left[\left\{\operatorname{erfc}\left(\frac{\varepsilon^{1/2}}{2\alpha^{1/2}}\right)+\operatorname{erfc}\left(-\frac{\varepsilon^{1/2}}{2\alpha^{1/2}}\right)\right.\right.$$
$$\left.\left.-\operatorname{erfc}\left(\alpha^{1/2}x-\frac{\varepsilon^{1/2}}{2\alpha^{1/2}}\right)\right\} e^{-x\varepsilon^{1/2}}+e^{x\varepsilon^{1/2}}\operatorname{erfc}\left(\alpha^{1/2}x+\frac{\varepsilon^{1/2}}{2\alpha^{1/2}}\right)\right]. \tag{10}$$

A zero-order fin model usually gives good results when the variation of the temperature in the lengthwise direction is moderate. This will be the case when the heat source is rather spread out (α small) and the heat losses to the surroundings are relatively small (ε small). This is illustrated by Table I which shows some results for α and ε both equal to 0.1. The differences between the exact results and the approximate ones are only of the order of a few percent. However, when either α or ε or both become larger, the zero-order fin model is likely to become less effective. Consequently, a need for more accurate approximate methods arises.

Higher-Order Fin Method

Instead of assuming the temperature to be constant across the slab, we shall assume that it may be written as

$$T=\sum_{n=0}^{N} a_n(x)\, y^{2n}, \tag{11}$$

where N is an integer that determines the order of the approximation. The odd

TABLE I

Performance of Zero-Order Fin Method for $\alpha = \varepsilon = 0.1$

x	Fin method	Exact ($y=0$)	Exact ($y=1$)
0	1 7255	1 6866	1 7551
1	1 6556	1 6216	1 6794
2	1 4680	1 4462	1 4782
3	1 2153	1.2073	1 2118
4	0 9539	0 9566	0 9424
5	0 7220	0 7307	0 7089
6	0 5350	0 5457	0 5241
7	0 3924	0 4029	0 3849
8	0 2865	0 2960	0 2821
9	0 2089	0 2170	0 2067
10	0 1523	0 1590	0 1514

powers of y have been left out because of Eq. (2). Now we derive a set of ordinary differential equations and one algebraic equation for the functions $a_n(x)$. First, we have from (3) and (11)

$$\sum_{n=0}^{N} (2n+\varepsilon)\, a_n = q(x). \tag{12}$$

N more equations are obtained when we integrate Eq. (1) up to N times across the slab in the following way

$$0 = \int_0^1 dy \int_0^y dy \cdots \int_0^y dy \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{13}$$

using Eq. (11). This yields N second-order differential equations

$$\frac{1}{i!}\frac{d^2 a_0}{dx^2} + \sum_{n=1}^{N} \frac{(2n)!}{(2n+i)!}\left\{\frac{d^2 a_n}{dx^2} + (2n+i-1)(2n+i)\, a_n\right\} = 0 \qquad (i = 1, 2, ..., N). \tag{14}$$

The boundary conditions are

$$\frac{da_n}{dx} = 0 \quad \text{at} \quad x = 0 \text{ for } n = 0, 1, ..., N \tag{15}$$

and

$$a_n \to 0 \quad \text{when} \quad x \to \infty \text{ for } n = 0, 1, ..., N. \tag{16}$$

Of course, these conditions have to be compatible with Eq. (12), which demands

$$q'(0) = 0 \quad \text{and} \quad q(\infty) = 0. \tag{17}$$

We shall illustrate the effectiveness of the higher order methods by referring to our earlier example with $q(x)$ as given by Eq. (6). The first-order method involves the equation

$$\frac{d^2a_0}{dx^2}+\frac{1}{3}\frac{d^2a_1}{dx^2}+2a_1=0 \tag{18}$$

and

$$\varepsilon a_0+(2+\varepsilon)\,a_1=\alpha^{1/2}e^{-\alpha x^2} \tag{19}$$

subject to the boundary conditions (15) and (16) for $n=0$ and $n=1$. The solution for a_0 can be expressed as

$$a_1(x)=\frac{\gamma\sigma}{4}\pi^{1/2}e^{\sigma^2/4\alpha}\left[e^{-\sigma x}\left\{-\operatorname{erfc}\left(-\frac{\sigma}{2\alpha^{1/2}}\right)-\operatorname{erfc}\left(\frac{\sigma}{2\alpha^{1/2}}\right)\right.\right.$$
$$\left.\left.+\operatorname{erfc}\left(x\alpha^{1/2}-\frac{\sigma}{2\alpha^{1/2}}\right)\right\}-e^{\sigma x}\operatorname{erfc}\left(x\alpha^{1/2}+\frac{\sigma}{2\alpha^{1/2}}\right)\right]+\gamma e^{-\alpha x^2}, \tag{20}$$

where

$$\sigma=\left(\frac{3\varepsilon}{3+\varepsilon}\right)^{1/2}\quad\text{and}\quad\gamma=\frac{3}{2(3+\varepsilon)}. \tag{21}$$

From (19) and (20) an expression for $a_0(x)$ may easily be derived, so that we can evaluate the temperature

$$T=a_0(x)+a_1(x)\,y^2. \tag{22}$$

The second-order model may be treated in exactly the same way, and an analytical solution may be found. It involves a little more algebra which we shall not present here in detail. Anyhow, in the application of the method to practical problems the ensuing system of differential equations cannot in general be solved analytically, and we shall have to resort to numerical integration.

For $\alpha=\varepsilon=1$ the second-order method means a considerable improvement upon the zero-order fin method. This is indicated by the results of Table II which demonstrate that the second-order method is very accurate across the board. We show in Table III how the methods perform for various values of α and ε. The values given in the table refer to the temperatures at $(x, y)=(0, 0)$ and $(x, y)=(0, 1)$, respectively. For all cases the approximate results seem to approach the exact values when methods of higher order are selected. Even in the case $\alpha=10$, $\varepsilon=10$, which means an extremely narrow heat source and very efficient heat transfer, both of which conditions are conducive to a rapid longitudinal temperature variation, the second-order method leads to a relatively good approximation. Although no

TABLE II

Comparison of Zero-, First-, and Second-Order Fin Methods with Exact Solution for $\alpha = 1$ and $\varepsilon = 1$

	$y=0$				$y=1$			
x	0	1	2	Exact	0	1	2	Exact
0.0	0.5456	0.4377	0.4552	0.4549	0.5456	0.6251	0.6139	0.6135
0.5	0.4927	0.4160	0.4241	0.4241	0.4927	0.5370	0.5328	0.5327
1.0	0.3707	0.3524	0.3470	0.3472	0.3707	0.3576	0.3627	0.3629
1.5	0.2459	0.2642	0.2564	0.2565	0.2459	0.2113	0.2166	0.2166
2.0	0.1531	0.1812	0.1773	0.1772	0.1531	0.1269	0.1285	0.1284
2.5	0.0933	0.1192	0.1181	0.1181	0.0933	0.0801	0.0797	0.0797
3.0	0.0567	0.0775	0.0775	0.0774	0.0567	0.0517	0.0510	0.0510

proof is presented here, it would seem that choosing methods of ever-increasing order, we may indeed obtain results that actually converge to the exact values.

Comparison with the Method of Lines

As explained in [8, Section 6.7], the method of lines also aims at reducing the dimensionality of a given boundary-value problem and solve it numerically. It does so by replacing the derivatives with respect to one of the coordinates by finite differences. A differential equation such as (1) is then replaced by a set of ordinary differential equations. In the model problem presented here the derivative to be replaced by finite differences would be $\partial^2 T/\partial y^2$. To arrive at a system *of the same complexity* as our higher order fin model we need to cover the y interval $[0, 1]$ by n mesh points. In addition to these, two virtual mesh points will have to be considered outside the interval at either side

In order to be able to compare this method with ours, we shall present the system that corresponds with the second-order fin model which involves two linear

TABLE III

Comparison of Zero-, First-, and Second-Order Fin Methods with Exact Solutions for Various α and ε

		$y=0$				$y=1$			
α	ε	0	1	2	Exact	0	1	2	Exact
1	0.3	1.2183	1.0777	1.1006	1.1001	1.2183	1.3719	1.3515	1.3510
1	3	0.2390	0.1710	0.1815	0.1813	0.2390	0.2684	0.2644	0.2643
0.3	1	0.4011	0.3567	0.3599	0.3599	0.4011	0.4204	0.4189	0.4189
3	1	0.6580	0.4486	0.5121	0.5095	0.6580	0.8764	0.8265	0.8234
10	1	0.7479	0.3494	0.5529	0.5379	0.7479	1.2870	1.1071	1.0851
10	10	0.1725	0.0233	0.0836	0.0775	0.1725	0.2674	0.2477	0.2433

second-order differential equations. To arrive at a system of only two differential equations when applying the method of lines, we can admit only two regular mesh points, viz. at $y=0$ and $y=1$, and two virtual mesh points at $y=-1$ and $y=2$. In addition to the two difference-differential equations obtained from Eq. (1)

$$T_0'' + T_1 - 2T_0 + T_{-1} = 0, \qquad T_1'' + T_2 - 2T_1 + T_0 = 0, \tag{23}$$

we have two difference equations that result from Eqs. (2) and (3)

$$2T_{-1} + 3T_0 - 6T_1 + T_2 = 0 \tag{24}$$

$$T_{-1} - 6T_0 + (3+6\varepsilon)\,T_1 + 2T_2 = 6\alpha^{1/2}e^{-\alpha x^2}. \tag{25}$$

Eliminating T_{-1} and T_2 from (23), (24), and (25) we obtain finally

$$T_0'' = 6T_0 - (6+2\varepsilon)\,T_1 + 2\alpha^{1/2}e^{-\alpha x^2} \tag{26}$$

$$T_1'' = -6T_1 + (6+4\varepsilon)\,T_1 - 4\alpha^{1/2}e^{-\alpha x^2}. \tag{27}$$

These equations will have to be solved subject to the boundary conditions

$$T_0'(0) = T_1' = T_0(\infty) = T_1(\infty) = 0. \tag{28}$$

The performance of the two methods is illustrated by Table IV. Clearly, the second-order fin method is superior for all values of α and ε presented. The results obtained by the method of lines are poor in comparison, given the same complexity of the two systems. Obviously, many more mesh points will be needed for the method of lines to yield results that are of comparable accuracy. But this would lead to an extra second-order ordinary differential equation for each additional mesh point, reducing the efficiency of the numerics. This is an illustration of the fact that higher order fin methods are likely to produce much faster computer codes than finite-difference methods of equal accuracy.

TABLE IV

Comparison of the Method of Lines with the Second-Order Fin Method (Numbers Given Are for $x=0$)

α	ε	y	Method of lines	Exact	Second-order fin
0.3	0.3	0	0 9206	0.9341	0 9342
0.3	0.3	1	1.0311	1.0432	1.0432
1	1	0	0 4320	0 4549	0 4552
1	1	1	0 5957	0 6135	0.6139
3	3	0	0 1773	0 2101	0 2119
3	3	1	0 3706	0.3928	0 3944

3. Extension to Higher Dimensions

To show that the method works equally well for problems of higher dimensionality, let us consider

$$D(T)=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}-\frac{\partial T}{\partial t}=0 \tag{29}$$

and describe time-dependent diffusion in a slab $0 \leqslant y \leqslant 1$. At $t=0$ we have $T=0$ everywhere in $-\infty < x < \infty$, $0 \leqslant y \leqslant 1$. For $t>0$ we consider the boundary condition

$$\frac{\partial T}{\partial y}=\alpha^{1/2} e^{-\alpha x^2}. \tag{30}$$

Furthermore, we have $\partial T/\partial y=0$ at $y=0$.

The solution to this problem can be derived by a combination of Laplace and Fourier transforms. Omitting details of the derivation, we have

$$T=\frac{1}{\pi^{1/2}}\int_0^\infty e^{-\omega^2/4\alpha}\cos(\omega x)\left[\frac{1-e^{-\omega^2 t}}{\omega^2}+\frac{1}{\omega}\left(\frac{\operatorname{ch}(\omega y)}{\operatorname{sh}\omega}-\frac{1}{\omega}\right)\right.$$
$$\left.-\,2e^{-\omega^2 t}\sum_{k=1}^{\infty}(-1)^k e^{-k^2\pi^2 t}\frac{\cos(k\pi y)}{\omega^2+k^2\pi^2}\right]d\omega. \tag{31}$$

Let us see how accurately the second-order fin method is able to approximate the values produced by this exact solution. The second-order fin method assumes

$$T=a_0(x,t)+a_1(x,t)\,y^2+a_2(x,t)\,y^4. \tag{32}$$

Substituting (32) in (29) and evaluating the following integrals

$$\int_0^1 D(T)\,dy \qquad \text{and} \qquad \int_0^1 dy\int_0^y D(T)\,dy, \tag{33}$$

we obtain

$$M(a_0)+\tfrac{1}{3}M(a_1)+\tfrac{1}{5}M(a_2)=2a_1+4a_2 \tag{34}$$

and

$$M(a_0)+\tfrac{1}{6}M(a_1)+\tfrac{1}{15}M(a_2)=2a_1+2a_2, \tag{35}$$

where

$$M(a)=\frac{\partial a}{\partial t}-\frac{\partial^2 a}{\partial x^2}. \tag{36}$$

The boundary condition at $y=1$ yields

$$2a_1+4a_2=\alpha^{1/2}e^{-\alpha x^2}. \tag{37}$$

The initial and boundary conditions read

$$t=0: a_0=a_1=a_2=0 \tag{38}$$

$$t>0: a_0\to 0, a_1\to 0, a_2\to 0 \quad \text{when} \quad x\to \pm\infty. \tag{39}$$

Using (37) we may recast the system of differential equations (34)–(35) into diagonal form

$$M(a_0)=f_0(a_0, a_1, x), \qquad M(a_1)=f_1(a_0, a_1, x), \tag{40}$$

which can easily be solved numerically by means of a tri-diagonal matrix operation. However, in this particular case it is also possible to find an analytic solution. Using the Laplace transform we obtain

$$a_0=\frac{1}{36}\alpha^{1/2}e^{-\alpha x^2}+\frac{7}{36}\pi^{1/2}\left\{e^{-10t}F\left(t+\frac{1}{4\alpha}, x\right)-F\left(\frac{1}{4\alpha}, x\right)\right\}$$

$$+G\left(\frac{1}{4\alpha}+t, x\right)-G\left(\frac{1}{4\alpha}, x\right) \tag{41}$$

$$a_2=\frac{5}{12}\alpha^{1/2}e^{-\alpha x^2}+\frac{5}{12}\pi^{1/2}\left\{e^{-10t}F\left(t+\frac{1}{4\alpha}, x\right)-F\left(\frac{1}{4\alpha}, x\right)\right\}, \tag{42}$$

where

$$F(\rho, x)=\frac{10^{1/2}}{4}e^{10\rho}\left[e^{-10^{1/2}x}\operatorname{erfc}\left\{(10\rho)^{1/2}-\frac{x}{2\rho^{1/2}}\right\}\right.$$

$$\left.+e^{10^{1/2}x}\operatorname{erfc}\left\{(10\rho)^{1/2}+\frac{x}{2\rho^{1/2}}\right\}\right] \tag{43}$$

and

$$G(\rho, x)=\rho^{1/2}e^{-x^2/4\rho}-\frac{\pi^{1/2}}{2}x\operatorname{erfc}\left(\frac{x}{2\rho^{1/2}}\right). \tag{44}$$

Some numerical results are listed in Tables Va,b. They refer to $x=0$, where the deviations of the fin method from the exact solution (31) are probably largest. Nevertheless, even these results are seen to be very accurate. This shows again that higher-order fin methods may be powerful alternatives for such well-known methods as the finite-element method or the finite-difference method. The latter two methods are known to become progressively more time consuming when applied to problems of higher dimensionality.

TABLE Va

Exact Solution of Problem of Section 3 ($x=0$)

y \ t	0.1	0.2	0.4	1.0	2.0	10.0
0.0	0.0052	0.0483	0.1678	0.4775	0.8595	2.5611
0.2	0.0117	0.0608	0.1831	0.4931	0.8751	2.5767
0.4	0.0346	0.1004	0.2300	0.5410	0.9230	2.6246
0.6	0.0837	0.1725	0.3121	0.6244	1.0064	2.7080
0.8	0.1755	0.2860	0.4351	0.7487	1.1307	2.8323
1.0	0.3332	0.4538	0.6073	0.9215	1.3034	3.0050

TABLE Vb

Results of Second-Order Fin Method Applied to Problem of Section 3 ($x=0$)

y \ t	0.1	0.2	0.4	1.0	2.0	10.0
0.0	0.0069	0.0489	0.1675	0.4770	0.8590	2.5605
0.2	0.0124	0.0612	0.1829	0.4929	0.8748	2.5764
0.4	0.0332	0.1000	0.2302	0.5413	0.9233	2.6248
0.6	0.0815	0.1718	0.3125	0.6250	1.0069	2.7085
0.8	0.1759	0.2860	0.4351	0.7487	1.1307	2.8323
1.0	0.3365	0.4544	0.6067	0.9208	1.3028	3.0043

4. A Fluid-Mechanical Example

Let us consider creeping flow though a straight infinitely long channel with a normalized width equal to unity. Fluid is injected into the channel through one of its side walls. To simplify the analysis, we shall assume the injection distribution to be Gaussian. An equal amount of fluid leaves the channel through the same side wall, but at a different location. Therefore, there is no fluid motion at infinity. In terms of the stream function we define the problem as follows:

$$\Delta\Delta\psi=0 \qquad (-\infty<x<\infty, 0\leqslant y\leqslant 1) \tag{45}$$

$$\psi=0 \quad \text{and} \quad \psi_y=0 \quad \text{at} \quad y=0 \tag{46}$$

$$\psi_y=0 \quad \text{at} \quad y=1 \tag{47}$$

$$\psi_x=e^{-(x+\alpha)^2}-e^{-(x-\alpha)^2} \tag{48}$$

$$\psi\to 0 \quad \text{when} \quad x\to -\infty \text{ or } x\to\infty. \tag{49}$$

In view of (46) we shall write the stream function as

$$\psi = y^2 \sum_{n=0}^{N} a_n(x)\, y^n, \tag{50}$$

where the functions $a_n(x)$ are to be determined by the fin method. We have from (47) and (48)

$$\sum_{n=0}^{N} (n+2)\, a_n = 0 \tag{51}$$

and

$$\sum_{n=0}^{N} a_n' = e^{-(x+\alpha)^2} - e^{-(x-\alpha)^2}. \tag{52}$$

Besides these two equations we need $N-1$ additional equations to be able to determine the unknown functions $a_n(x)$. These are obtained through multiple integrations of the kind discussed before. These lead to

$$\sum_{n=0}^{N} \frac{(2n+1)!}{(2n+i+1)!} a_n^{\mathrm{IV}} + 2 \sum_{n=1}^{N} \frac{(2n+1)!}{(2n+i-1)!} a_n'' + \sum_{n=2}^{N} \frac{(2n+1)!}{(2n+i-3)!} a_n = 0$$
$$(i = 1, 2, \ldots, N-1). \tag{53}$$

Clearly, the method is only meaningful if N is at least equal to two. The boundary conditions for the functions $a_n(x)$ follow directly from Eqs. (49) and (50)

$$a_n \to 0 \quad \text{when} \quad x \to -\infty \text{ or } x \to \infty. \tag{54}$$

In general, a system of equations such as (51)–(54) will have to be solved numerically by means of a shooting method or by direct discretization. However, the particular system (51)–(54) can be solved more easily by the application of the Fourier transform

$$A_n(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} a_n(x)\, dx. \tag{55}$$

This leads to

$$\sum_{n=0}^{N} (n+2)\, A_n = 0 \tag{56}$$

$$\sum_{n=0}^{N} A_n = 2\pi^{1/2} e^{-\omega^2/4} \frac{\sin \omega\alpha}{\omega} \tag{57}$$

$$\omega^4 \sum_{n=0}^{N} \frac{(2n+1)!}{(2n+i+1)!} A_n - 2\omega^2 \sum_{n=1}^{N} \frac{(2n+1)!}{(2n+i-1)!} A_n + \sum_{n=2}^{N} \frac{(2n+1)!}{(2n+i-3)!} A_n = 0$$
$$(i = 1, 2, \ldots, N-1). \tag{58}$$

For each given value of N, Eqs. (56)–(58) constitute a set of exactly N algebraic equations for the transformed variables A_n. This set may be solved, and the A_n may be transformed backwards to produce the functions $a_n(x)$

$$a_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} A_n(\omega)\, d\omega. \tag{59}$$

For $N = 2$ we find

$$a_0(x) = \frac{8}{\pi^{1/2}} \int_0^{\infty} \frac{180 - 6\omega^2 + \frac{1}{2}\omega^4}{240 + 8\omega^2 + \frac{2}{3}\omega^4} e^{-\omega^2/4} \frac{\sin \omega\alpha}{\omega} \cos \omega x \, d\omega \tag{60}$$

$$a_1(x) = 2\pi^{1/2}\{2 - \text{erfc}(\alpha - 2) - \text{erfc}(\alpha + 2)\} - 2a_0(x) \tag{61}$$

$$a_2(x) = -\tfrac{1}{2}a_0(x) - \tfrac{3}{4}a_1(x). \tag{62}$$

The integral occurring in (60) can be solved rapidly by means of some numerical quadrature.

TABLE VI

Position of Central Streamline as Calculated by a Succession of Higher Order Fin Methods

	y			
x	$N=2$	$N=3$	$N=4$	$N=5$
0 000	0 56946324	0 56888542	0 56887509	0 56887565
0 100	0 57137792	0 57080810	0 57079605	0.57079657
0 200	0 57720629	0 57666090	0 57664404	0 57664444
0 300	0 58721395	0.58671050	0 58668675	0 58668693
0 400	0 60188985	0 60144708	0 60141592	0 60141580
0 500	0 62203563	0 62167280	0 62163573	0 62163525
0 600	0 64894864	0 64868348	0 64864421	0.64864335
0 700	0 68482880	0 68467348	0 68463775	0 68463662
0 800	0 73386264	0 73381642	0.73379113	0 73378994
0 900	0 80632925	0 80636448	0 80635524	0 80635439
0 920	0 82599725	0 82604110	0 82603522	0.80603449
0 940	0 84875915	0 84880739	0 84880464	0.84880405
0 960	0.87622929	0 87627632	0.87627624	0.87627580
0 970	0 89277965	0 89282326	0 89282421	0 89282385
0 980	0 91251596	0 91255352	0 91255518	0.91255492
0 990	0 93830268	0.93833013	0 93833195	0 93833179
0 992	0 94487080	0.94489542	0 94489717	0 94489703
0 994	0 95231772	0 95233904	0 95234067	0 95234055
0 996	0 96113499	0 96115235	0 96115377	0 96115367
0 998	0.97259005	0 97260223	0 97260329	0 97260323

Note Problem is symmetric about $x = 0$.

Clearly, when N becomes larger, the algebra becomes more awkward. The explicit solution of the functions $A_n(x)$ as functions of ω becomes cumbersome. Of course, formula-manipulation routines such as REDUCE or MACSYMA may be of help here. We have found that solving the algebraic equations for each value of ω required by the numerical integration routine is also an efficient and easy-to-apply way of solving the higher order problems.

Results are shown in Table VI for $N = 2, 3, 4, 5$. The numbers in the table refer to the position of the streamline which runs between the centers of the source and the drain. We may conclude from these results that the higher order fin method is very effective in fluid mechanics also. By the same token applications in elasticity can be envisaged. It is reasonable to assume that the method can be applied to more challenging fluid-mechanical problems, such as the flow through a continuously expanding or contracting conduit. However, such an application seems to be outside the scope of the present paper which aimed at presenting some initial ideas only.

5. A Nonlinear Example

The examples we presented up to now were all linear. However, the fin-model concept can also be applied to nonlinear problems. Indeed, it is precisely here that we expect these methods to be of great help. Whereas the linear problems that we put forward could all be solved analytically, nonlinear problem do not easily produce analytical solutions. Therefore, we shall give a nonlinear example for which an analytical solution can be obtained artificially.

Let us consider the heat-transfer problem of Section 2. However, instead of the linear radiation boundary condition (3) we use

$$\frac{\partial T}{\partial y} = -\varepsilon(T + T^4) + q(x). \tag{63}$$

The function $q(x)$ still remains to be selected. The solution to our problem shall be given by Eq. (7) with $\alpha = 1$. Let us denote this function by $f(x, y)$. The function $f(x, y)$ satisfies Eqs. (1), (2), (4), and (5). For it to satisfy Eq. (63), we must have

$$q(x) = q_0(x) + \varepsilon f^4(x, 1), \tag{64}$$

where $q_0(x)$ is given by Eq. (6) with $\alpha = 1$.

Zero-Order Fin Method

The requisite equation follows immediately from Eqs. (8) and (63):

$$\frac{d^2T}{dx^2} - \varepsilon(T + T^4) = -q. \tag{65}$$

The boundary conditions are those given by Eqs. (4) and (5).

First-Order Fin Method

Next we write $T=a_0(x)+a_1(x)\,y^2$. Using (12) and (13) we obtain Eq. (18) and

$$2a_1+\varepsilon\{a_0+a_1+(a_0+a_1)^4\}=q(x) \tag{66}$$

with boundary conditions as given by Eqs. (15) and (16). Writing

$$A_0=a_0+a_1, \qquad A_1=a_1, \tag{67}$$

and substituting this in the requisite equations, we may eliminate A and obtain a differential equation involving the function A only:

$$\left\{1+\frac{1}{3}\varepsilon(1+4A_0^3)\right\}\frac{d^2A_0}{dx^2}+4\varepsilon\left(A_0\frac{A_0}{dx}\right)^2-\varepsilon(A_0+A_0^4)=\frac{1}{3}\frac{d^2q}{dx^2}-q. \tag{68}$$

The boundary conditions are

$$\frac{dA_0}{dx}=0 \quad \text{at } x=0, \qquad A_0\to 0 \quad \text{when } x\to\infty \tag{69}$$

Second-Order Fin Method

We shall carry our analysis one step further by writing

$$T=a_0(x)+a_1(x)\,y^2+a_2(x)\,y^4. \tag{70}$$

Introducing

$$A_0=a_0+a_1+a_2, \qquad A_1=a_1, \qquad A_2=a_2, \tag{71}$$

we may carry out an elimination process such as that presented above for the first-order case. This results in

$$\left\{\frac{3}{2}+\frac{1}{6}\varepsilon(1+4A_0^3)\right\}\frac{d^2A_0}{dx^2}=\varepsilon\left\{\frac{7}{2}A_0+\frac{7}{2}A_0^4-2\left(A_0\frac{dA_0}{dx}\right)^2\right\}$$
$$+\frac{1}{6}\frac{d^2q}{dx^2}-\frac{7}{2}q+4A_1 \tag{72}$$

$$\frac{d^2A_1}{dx^2}=\frac{15}{4}\left[\left\{1+\frac{1}{5}\varepsilon(1+4A_0^3)\right\}\frac{d^2A_0}{dx^2}\right.$$
$$\left.+\varepsilon\left\{\frac{12}{5}\left(A_0\frac{dA_0}{dx}\right)^2-A_0-A_0^4\right\}+q-\frac{1}{5}\frac{d^2q}{dx^2}\right]. \tag{73}$$

Of course, by differentiating (72) twice with respect to x, we may substitute Eq. (73)

TABLE VII

Temperature at Selected Locations in the Plane $y = 1$ as Determined by the Nonlinear Problem of Section 5 ($\varepsilon = 1$)

	Order of fin method			
x	0	1	2	Exact
0.0	0.5597	0.6218	0.6138	0.6135
0.5	0.5037	0.5355	0.5327	0.5327
1.0	0.3770	0.3573	0.3627	0.3629
1.5	0.2494	0.2110	0.2166	0.2166
2.0	0.1552	0.1267	0.1285	0.1284
2.5	0.0946	0.0800	0.0797	0.0797
3.0	0.0574	0.0516	0.0510	0.0510

and arrive at a fourth-order differential equation for A_0. The boundary conditions are

$$\frac{dA_0}{dx} = 0 \quad \text{and} \quad \frac{dA_1}{dx} = 0 \quad \text{at} \quad x = 0 \tag{74}$$

$$A_0 \to 0 \quad \text{and} \quad A_1 \to 0 \quad \text{when} \quad x \to \infty. \tag{75}$$

Results

The results collected in Table VII show that fin methods also work satisfactorily when they are applied to nonlinear problems. Indeed, comparing these results with those listed in the right-hand section of Table II, we notice that the performance of the methods is as good as for the linear case.

6. Concluding Remarks

In this paper we have extended the traditional fin-method approach to higher order fin methods. It was shown that the method can easily be applied to problems involving more than two independent variables. It can also readily be envisaged that problems involving more than one dependent variable will be equally suitable for treatment by higher order fin methods. The numerical examples show that fin methods of second order already are capable of producing results that are accurate, even in those cases where the field variable varies rapidly.

Although fin methods were originally designed to deal with heat-transfer problems, the nature of the method does not seem to preclude application in fields that are quite different from heat transfer. For instance, low-Reynolds-number flow in a not-so-slowly expanding or contracting conduit seems to be an example where these higher order fin methods may yield good results at a relatively low cost.

In the examples that we showed, the boundary conditions (e.g., heat-input function $q(x)$) were smooth. No doubt, the excellent performance of the method has to be attributed to these smooth boundary conditions. It is to be expected that the method will work less well when the boundary conditions change abruptly, and particularly in the neighborhood of these discontinuities. The assumption that a truncated Taylor-series expansion in y (Eq. (11)) is able to model the field is obviously at variance with reality in those cases. It can be expected, however, that at some distance from these discontinuities the accuracy of the method will improve.

It should be realized that the methods proposed here do not merely entail the substitution of a truncated series expansion in all constituent elements of the problem at hand. Such an approach would require a grouping of like powers of the expansion variable. This would necessarily lead to the dropping of entered terms at the higher end. It is expected that this will affect the accuracy of the solution, particularly in the case of nonlinear problems, where higher order terms that do not appear in the truncated series may crop up after substitution. Our multiple-integral rule, e.g., Eq. (13), ensures that these terms, which would otherwise not be counted, remain an integral part of the solution.

Most of the examples given in this paper were linear boundary-value problems. The reason for this was, of course, that in these cases analytical solutions could easily be found which would serve as test cases for our numerical procedures. However, there is no reason to believe that higher order fin methods will perform less efficiently for nonlinear problems. Results we obtained for a nonlinear example support this belief. The only requirement for these methods to be useful tools is that the field variable(s) should be (a) reasonably smooth function(s) of the independent variables. As such, higher order fin methods can be considered as easy-to-use alternatives for Galerkin or Rayleigh–Ritz methods.

Finally, methods such as the ones presented here are reminiscent of methods that were already proposed in the pre-computer days, particularly in the Russian sphere of thought. The book by Holt [8], which we mentioned already, strongly advocates these methods and shows that they may produce accurate results. Fin methods are also akin to the method of weighted residuals discussed in [9]. In that book the basic functions are forced to satisfy the boundary conditions. Fin methods seem to work very well with the simplest of basic functions (powers) and add an equation such as (12) to take care of the boundary condition(s). What all these methods seem to show is that very often it is quite unnecessary to apply involved finite-difference or finite-element schemes to problems that can be solved equally accurately at a much reduced cost Moreover, many of the geometries with sharp corners, where these methods are less efficient, are undesirable from a technological point of view, particularly in fluid mechanics and elasticity. These sharp features produce either unwanted eddies or much increased local stress levels.

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REFERENCES

1. D Q KERN AND A D KRAUS, *Extended Surface Heat Transfer* (McGraw–Hill, New York, 1972)
2 A AZIZ AND T Y NA, *Perturbation Methods in Heat Transfer* (Hemisphere, Washington, 1984), pp. 14, 44, 140
3 F M WHITE, *Heat Transfer* (Addison–Wesley, Reading, Ma, 1984), p 70
4. H. K KUIKEN, *J Eng. Math* **13**, 97 (1979).
5. H K KUIKEN AND P. J ROKSNOER, *J Cryst. Growth* **47**, 29 (1979)
6 T JASINSKI, W M ROHSENOW, AND A F WITT, *J. Cryst Growth* **61**, 339 (1983)
7 C E CHANG AND W R WILCOX, *J Cryst. Growth* **21**, 135 (1974).
8 M HOLT, *Numerical Methods in Fluid Dynamics* (Springer, Berlin, 1977), pp 1, 212
9 B A FINLAYSON, *The Method of Weighted Residuals and Variational Principles* (Academic Press, New York, 1972), pt. I.